Chapter 3 Notes

1. Unknotting Number

(1) Definition: A knot K has unknotting number n if there exists a projection of the knot such that changing n crossings in the projection turns the knot into the unknot and there is no projection such that fewer changes would have turned it into the unknot .

(2) Notation: u(K).

(3) The fact that every knot has a finite unknotting number follows from the fact that every projection of a knot can be changed into a projection of the unknot by changing some subset of the crossings in the projection.

(4) **Cyclic Surgery Theorem**: certain types of three-manifolds can be obtained from a given knot complement by performing a specific type of surgery operation. In topology, surgery refers to a process of cutting a manifold into smaller pieces, and then reassembling them in a different way to obtain a new manifold. If a knot complement is modified in this way, the resulting manifold will have the same topological properties as the original knot complement, up to a certain degree of freedom.

(5) A knot with u(K) = 1 is prime.

(6) The unknotting number of a knot can be realized by a non-minimal projection.

(7) **k-equivalent**: when two knots or links can be transformed into each other through a series of k-moves and -k-moves (Reidemeister moves). For example, two knots are 0-equivalent if they are the same knot, and they are 1-equivalent if they can be transformed into each other with a single Reidemeister move.

2. Bridge Number

(1) Definition: the least bridge number of all the projections of the knot.

(2) Notation: b(k).

(3) **Overpass**: a subarc of the knot that goes over at least one crossing but never goes under a crossing.

(4) **Maximal overpass**: an overpass that could not be made any longer.

(5) **Two-bridge knots**: knots with b(K) = 2. They are also prime knots.

(6) The number of distinct two-bridge knots of n crossings is at least $(2^{n-2}-1)/3$.

3. Crossing Number

(1) Definition: the least number of crossings that occur in any projection of the knot.

(2) Notation: c(K).

(3) **Reduced**: projection of a knot where there are no easily removed crossings.